Topics in percolation theory

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Summary

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The topics treated in the thesis are centered around the percolation model. It is a model for porous rock, where the holes of the rock are described by a regular structure of interconnecting tubes as follows. Let $p \in [0, 1]$ be the parameter of the model. Each tube, independently from each other, is open with probability p, that is, it allows water to pass through, or it is closed otherwise. Similarly to other models of statistical mechanics, we are mainly interested the large scale (macroscopic) properties of the model. That is, we concentrate on the case where the tubes are much smaller than the rock itself. It turns out that there is a critical parameter $p_c \in [0, 1]$ such that when $p < p_c$, there are only small holes. However, when $p > p_c$, the rock resembles a sponge: if we put the rock in water, the water percolates to the middle of the rock.

The phenomenon described above is the so called phase transition. An other example for this phenomenon is water at different temperatures: below 0 Celsius we have ice, while above 0 Celsius, we get liquid water. Consider the percolation model in a broader context:

We see that different small scale properties (the probability that a tube is open) can result in different large-scale behavior (whether the water penetrates the middle of the rock or not). Hence, the percolation model supports the idea that atomic scale properties of the matter can determine its state, i.e whether it is gas, liquid or solid. Another example supporting this idea is the more realistic and closely related Ising model for magnetism.

The properties of the percolation model described above make the model worthy of further investigation. It is particularly interesting at the critical point $p = p_c$. Substantial progress has been made in the last two decades on the critical model on two dimensional lattices. By combining results complex and stochastic analysis, geometry and graph theory, some remarkable symmetries of the two dimensional critical model were deduced. Nevertheless, still numerous easy to state problems are left open.

The aim of the thesis is to contribute to the theory of percolation. We refine some of the tools used in the study of the model, and investigate some related models. We first turn to the description of some of the tools used.

Recall the lines above where we described the phase transition of the percolation model. A bit more detailed analysis gives that when the tubes are much smaller than the rock, we see a so called approximate $0-1$ law. For some small $\varepsilon > 0$ we have that for $p > p_c + \varepsilon$ with probability close to 1, the water penetrates the middle of the rock, while for $p < p_c - \varepsilon$, the probability of this event is close to 0. As it turns out, this approximate $0-1$ law follows from some concentration inequalities and sharp threshold results. These results describe some fundamental properties of functions of many independent random variables. Hence they are used in a variety of disciplines such as in computer science for the analysis of randomized algorithms, or in economics for studying voting models. We motivate our results through an example of a voting model. Let us assume that we have n voters each and each of them, independently from each other, decides 'yes' or 'no' with probability $1/2$. There is decision scheme which tells what is the outcome of the vote given the decisions of the voters. The voting power (influence) of an individual i is the probability that the outcome of the vote changes if the individual i changes his decision, while the decisions of the other voters is left unchanged. It is a celebrated result that, roughly speaking, if the voting power of each individual is small, then the sum of the voting powers is large. Our contribution to this field is a a generalization of Talagrand's inequality which is a quantified version of the result above. Our result is, strictly speaking, not new, but our proof is different from the those in the literature.

Next, we investigate the first passage percolation model. It can be seen as an extension of the percolation model: instead of declaring each tube open or closed, we assign a positive number called the passage time. It represents the time it takes for the water to pass through the tube. We investigate how the water propagates inside the rock when we supply it at a given position. We give an upper bound on the variance of the speed of the propagating liquid. Bounds analogous to ours had already appeared in the literature in the case where the passage times of different tubes are independent from each other. The novelty of our method is that it extend the some of results in the literature to the case where the passage times are weakly dependent. Our generalized version of Talagrand's inequality plays a crucial role in the proof our variance bound above.

In the first passage percolation model above, the wetted area grows as we increase time. In the following we consider another growth model. The starting point is the percolation model, where the regular structure of tubes is some infinite system, such as a binary tree or a lattice.

Our starting point is the version of the percolation model where at time 0 all the tubes are closed. Then each tube, independently from each other, opens up at some random time which is uniformly distributed on the interval $[0, 1]$.

We get the following: at time $p = 0$ every tube is closed. Then as we increase p, then tubes open up, and start forming bigger and bigger holes (open clusters). For $p > p_c$, some infinite open clusters emerge, and at time $p = 1$ all the tubes are open. We modify this process by forbidding open clusters with size larger than N to grow any further, where N is the parameter of the model. Hence open clusters with size larger than N 'freeze'. We investigate the large N behavior of this so-called N-parameter frozen percolation process for the cases where the regular structure of the tubes is the binary tree and the two dimensional square lattice. Finally, we note that the there is a hidden parameter in the model: the way we measure the size of the clusters.

Let us turn to the process on the binary tree. We show that as N tends to ∞, under some mild conditions on the way we measure the size of the clusters, the N-parameter frozen percolation processes on the binary tree converge to the process where we replace N by ∞ in the description above. The dynamics of this ∞-parameter process drives the model to a state which at times at times larger than $1/2$ resembles the critical percolation model on the binary tree. This is an example of the so-called self-organized critical phenomenon, and makes the model quite interesting: Other examples for this phenomenon are believed to be earthquakes and fluctuations in financial markets.

The situation is quite different for the case of the square lattice, since the ∞-parameter process does not exists. Moreover, the behavior of the N-parameter frozen percolation process does depend on the way we measure the clusters. For this reason, we restrict to the case where the clusters are measured by their diameter. It turns out, as N tends to ∞, all of the frozen clusters form only at times close to $p_c = 1/2$, and are similar to critical percolation clusters. Furthermore, our results provide precise bounds on the times at which the frozen clusters form. This brings us to a conjecture, which roughly speaking, states that when look at the N-parameter frozen percolation process close to time $1/2$, scale space by N and time using the bounds above, we get a limiting process. We also believe that this process completely describes the large N behavior of the N-parameter processes. Proving this conjecture and investigating the N-parameter processes in the case where we measure the size of the clusters in a different way are challenges left for future research.